# Signal Analysis & Communication ECE 355

## Ch. 2.3: Properties of LTI Systems

Lecture 10

28-09-2023

# Ch. 2·3: PROPERTIES OF LTI SYSTEMS (Contd.)

## ④ LTI Memory

**Recall:** A system is memoryless iff for all time '$t$', the output $y_{(t)}$ depends only on the input $x_{(t)}$ at that same time.

- We know the LTI sys. is characterized by impulse response,

$$y_{(t)} = \int_{-\infty}^{\infty} x_{(\tau)}\, h_{(t-\tau)}\, d\tau$$

- Now the memorylessness implies that

$$h_{(t-\tau)} = 0 \qquad \text{for all } \tau \neq t$$

$$\text{OR} \quad h_{(t)} = 0 \qquad \text{for } t \neq 0$$

Sys. response is zero for all the non-present times.

$$\Rightarrow \quad y_{(t)} = \int_{-\infty}^{\infty} x_{(t)}\, h_{(t-\tau)}\, d\tau$$

$$= x_{(t)} \underbrace{\int_{-\infty}^{\infty} h_{(t-\tau)}\, d\tau}$$

We consider LTI Systems with finite impulse response (FIR)

$$k = \int_{-\infty}^{\infty} h_{(\tau)}\, d\tau$$

$$y_{(t)} = k\, x_{(t)} \qquad \text{— ①}$$

- Consider

$$x_{(t)} = \delta_{(t)} \quad \text{to an LTI}$$

$$\text{eqn ①} \Rightarrow \quad y_{(t)} = k\, \delta_{(t)}$$

Also, the O/P of the sys. when I/P is $\delta_{(t)}$, is $h_{(t)}$

$$\boxed{h_{(t)} = k\, \delta_{(t)}}$$

$$x_{(t)} = \delta_{(t)} \longrightarrow \boxed{S} \longrightarrow y_{(t)} = h_{(t)}$$

**THEOREM** A CT LTI system with impulse response $h_{(t)}$ is memoryless if and only if

$$h_{(t)} = k\, \delta_{(t)}$$

$$\text{DT LTI:} \quad h_{[n]} = k\, \delta_{[n]} \qquad \text{for some } k \in \mathbb{C}$$

## ⑤ LTI Invertibility

**Recall:** A system is invertible if there exists another system such that

$$T_{inv}\{\underbrace{T\{x(t)\}}_{y(t)}\} = x(t)$$

$$x(t) \longrightarrow \boxed{T} \xrightarrow{\;y(t)\;} \boxed{T_{inv}} \longrightarrow x(t)$$

**THEOREM** If an LTI system is invertible, then it's inverse system is also LTI.

**Proof**

Suppose $$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{\;y(t)\;} \boxed{T_{inv}} \longrightarrow x(t)$$

a) Linearity implies that

If $\begin{cases} x_1(t) * h(t) = y_1(t) \\ x_2(t) * h(t) = y_2(t) \end{cases}$ $\quad \begin{matrix} (\& \; y_1(t) \xrightarrow{T_{inv}} x_1(t)) \\ (\& \; y_2(t) \xrightarrow{T_{inv}} x_2(t)) \end{matrix}$

Then
$$[a_1 x_1(t) + a_2 x_2(t)] * h(t) = a_1 y_1(t) + a_2 y_2(t)$$

Hence
$$a_1 y_1(t) + a_2 y_2(t) \xrightarrow{T_{inv.}} a_1 x_1(t) + a_2 x_2(t)$$

b) Time-invariancy implies that

$$x(t-t_0) * h(t) = y(t-t_0)$$

We have $\quad y(t-t_0) \xrightarrow{T_{inv.}} x(t-t_0) \quad \left[\text{as } y(t) \xrightarrow{T_{inv.}} x(t)\right]$

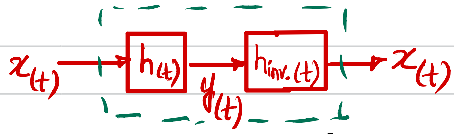
**LEMMA** A CT LTI system with impulse response $h(t)$ is invertible if and only if there exists another impulse response $h_{inv.}(t)$ such that

$$\boxed{h(t) * h_{inv.}(t) = \delta(t)}$$

**Proof** For a CT LTI:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

- To have $x(t)$ back




$$x(t) = \int_{-\infty}^{\infty} x(\tau) \underbrace{\left[ h(t) * h_{inv.}(t) \right]}_{\text{Should be } \delta(t-\tau)} d\tau$$

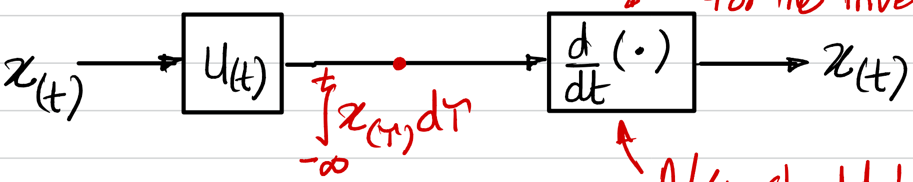
$$= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= x(t)$$ using sifting property of $\delta(t)$

**Example:**

$h(t) = u(t)$, $h_{inv.}(t) = ?$

since integral of $x(t)$ is I/P for the inverse it has to be differential




Also should be LTI characterized by impulse response

$$h_{inv.}(t) = \frac{d}{dt}\left(\delta(t)\right)$$ ← Response when unit impulse is applied.

$$h(t) * h_{inv.}(t) = h_{inv.}(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h_{inv.}(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h_{inv.}(\tau)\, u(t-\tau)\, d\tau.$$

$$= \int_{-\infty}^{t} h_{inv.}(\tau)\, d\tau = \int_{-\infty}^{t} \frac{d}{d\tau}\left(\delta(\tau)\right) d\tau.$$

$$= \delta(t)$$

DT LTI: Similar

**Example**

$$h[n] = u[n], \quad h_{inv.}[n] = ?$$

$$x[n] \longrightarrow \boxed{u[n]} \xrightarrow{\;y[n] = \sum_{k=-\infty}^{n} x[k]\;} \boxed{h_{inv.}[n]} \longrightarrow x[n]$$

$\delta[n] - \delta[n-1]$ should be the impulse response.

$$h_{inv.}[n] = \delta[n] - \delta[n-1]$$

to get $x[n] = y[n] - y[n-1]$

$$= \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k]$$

Now,

$$h[n] * h_1[n] = u[n] * (\delta[n] - \delta[n-1])$$

$$= \sum_{k=-\infty}^{n} \delta[k] - \sum_{k=-\infty}^{n-1} \delta[k]$$

$$= u[n] - u[n-1]$$

$$= \delta[n]$$

Relation b/w $\delta[n]$ & $u[n]$

⑥ **LTI Causality**

**Recall:** A system is causal if for all time, the output $y(t)$ depends only on the past & present values of the input, $\{x(\tau)\}_{\tau \le t}$.

- For CT LTI systems, the input output relation is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

- The causality in CT LTI implies that

$$h(t-\tau) = 0 \qquad \text{for } \tau > t$$

OR

$$h(t) = 0 \qquad \text{for } t < 0$$

Negative argument of h!

to make sure $y(t)$ depends only on $x(\tau)$ for $\tau < t$

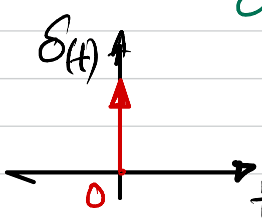
**THEOREM:** A CT LTI system with impulse response $h(t)$ is causal if & only if

$$\boxed{h(t) = 0, \quad \forall t < 0}$$

i.e., if $h(t)$ is right-sided

* Intuition:

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$




✓ The input has action only at $t=0$

✓ & Causality implies that the O/P $h(t)$ must be zero before $t=0$.

DT LTI: Similar

Example

$$h(t) = u(t) \qquad (u(t) = 0, \text{ for } t < 0)$$

$$h[n] = u[n] \qquad (u[n] = 0, \text{ for } n < 0)$$

NOTE:

① Causality for LTI is equivalent to the initial rest condition (Prob. 1.44), that is,

$$\text{if } x(t) = 0 \quad \text{for } t < t_0$$

$$\text{then } y(t) = 0 \quad \text{for } t < t_0$$

② If an LTI is causal then $[h(t) = 0 \text{ for } t < 0]$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y_{(t)} = \int_{-\infty}^{t} x_{(\tau)} h_{(t-\tau)} d\tau \text{ — } ② \quad (\text{as } h_{(t-\tau)} = 0 \text{ for } \tau > t)$$

$$= \int_{\infty}^{0} h_{(s)} x_{(t-s)} (-ds)$$

By assuming $s = t - \tau$
$\Rightarrow ds = -d\tau$
$\Rightarrow$ limits: $\infty \to 0$

$$= \int_{0}^{\infty} h_{(s)} x_{(t-s)} ds$$

$$= h_{(t)} * x_{(t)}$$ with limit of conv. integral as $0 \to \infty$.

③ Sometimes a sig. $x_{(t)}$ is called causal if $x_{(t)} = 0$ for $t < 0$ — Right sided sig.

④ For right sided input sig., i.e., $x_{(t)} = 0$ for $t < 0$, the output for causal LTI is

$$y_{(t)} = \int_{0}^{t} x_{(\tau)} h_{(t-\tau)} d\tau$$

(from eqn ② above).